

How to Calculate the Exponential of Matrices

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Abstract

How to calculate the exponential of matrices in an explicit manner is one of fundamental problems in almost all subjects in Science.

Especially in Mathematical Physics or Quantum Optics many problems are reduced to this calculation by making use of some approximations whether they are appropriate or not. However, it is in general not easy.

In this paper we give a very useful formula which is both elementary and getting on with computer.

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To calculate the exponential of matrices

$$e^A = \sum_{m=0}^{\infty} \frac{A^m}{m!} \quad \text{for } A \in M(n; \mathbf{C}) \quad (1)$$

is one of fundamental problems in almost all subjects in Science ¹.

In fact, in Mathematical Physics or Quantum Optics many problems are reduced to this calculation by use of some approximations. However, this is a very hard problem. See, for example, hard and “painful” calculations in [2] and [3].

In usual textbooks of Linear Algebra (see for example [4]) it is recommended to diagonalize A like

$$A = U D_A U^{-1} \quad \text{for some } U \in GL(n; \mathbf{C}) \quad (2)$$

where D_A is the diagonal matrix consisting of all eigenvalues of A . Unfortunately, this method is not realistic as known well.

Let us explain in more detail. First of all we write the characteristic equation (polynomial) of A :

$$0 = |\lambda E - A| = \lambda^n + p_1 \lambda^{n-1} + \cdots + p_{n-1} \lambda + p_n \quad (3)$$

where $p_1 = -\text{tr} A$ and $p_n = (-1)^n \det(A)$. (3) can be decomposed into

$$\lambda^n + p_1 \lambda^{n-1} + \cdots + p_{n-1} \lambda + p_n = (\lambda - \alpha_1)(\lambda - \alpha_2) \cdots (\lambda - \alpha_n) \quad (4)$$

where $\alpha_j \in \mathbf{C}$. From this we have

$$\begin{aligned} p_1 &= - \sum_{j=1}^n \alpha_j \\ p_2 &= (-1)^2 \sum_{i < j}^n \alpha_i \alpha_j \\ &\vdots \\ p_n &= (-1)^n \prod_{j=1}^n \alpha_j \end{aligned} \quad (5)$$

¹See the dictionary [1] concerning several mathematical notations in the paper

For the eigenvalue α_j ($j = 1, 2, \dots, n$) we define $|\alpha_j\rangle \in \mathbf{C}^n$ to be the eigenvector which is not necessarily normalized ²

$$A|\alpha_j\rangle = \alpha_j|\alpha_j\rangle, \quad (6)$$

then we obtain the diagonalization (2) if we define U as

$$U = (|\alpha_1\rangle, |\alpha_2\rangle, \dots, |\alpha_n\rangle) \in GL(n; \mathbf{C}) \quad (7)$$

and $D_A = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$.

Weak points of this method are that

(a) to find the eigenvectors (6) explicitly,

(b) to calculate U^{-1} the inverse of U .

They become more and more difficult as n becomes large.

Now, it is better to change the strategy. The famous theorem of Cayley–Hamilton gives

$$A^n + p_1 A^{n-1} + \dots + p_{n-1} A + p_n E = 0 \implies A^n = -p_1 A^{n-1} - \dots - p_{n-1} A - p_n E \quad (8)$$

, so we can formally write

$$e^A = \sum_{m=0}^{\infty} \frac{A^m}{m!} = f_0 E + f_1 A + f_2 A^2 + \dots + f_{n-1} A^{n-1} \quad (9)$$

where $f_j = f_j(p_1, p_2, \dots, p_n)$ for simplicity. **The purpose in the following is to determine $\{f_0, f_1, \dots, f_{n-1}\}$ explicitly ³.**

Here we use the notation like

$$\begin{aligned} & f_0 E + f_1 A + f_2 A^2 + \dots + f_{n-1} A^{n-1} \\ &= E f_0 + A f_1 + A^2 f_2 + \dots + A^{n-1} f_{n-1} \equiv (E, A, A^2, \dots, A^{n-1}) \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{pmatrix}. \end{aligned} \quad (10)$$

²In general, to normalize a system of vectors is not easy. See the example in the latter half.

³This way of thinking is very natural because A is finite-dimensional

Note that

$$A^n = (E, A, A^2, \dots, A^{n-1}) \begin{pmatrix} -p_n \\ -p_{n-1} \\ \vdots \\ -p_2 \\ -p_1 \end{pmatrix}$$

from (8).

From (3) we define the matrix

$$L = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -p_n \\ 1 & 0 & 0 & \cdots & 0 & -p_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -p_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -p_2 \\ 0 & 0 & 0 & \cdots & 1 & -p_1 \end{pmatrix}, \quad (11)$$

which is called the companion matrix. It is well known that L also satisfies the same characteristic equation (3)

$$0 = |\lambda E - L| = \lambda^n + p_1 \lambda^{n-1} + \cdots + p_{n-1} \lambda + p_n. \quad (12)$$

We reconsider an important role that L plays.

Now, we are in a position to state the main result.

Fundamental Lemma For any $m \in \mathbf{N} \cup \{0\}$ we have

$$A^m = (E, A, A^2, \dots, A^{n-1}) L^m \mathbf{e}_1 \quad (13)$$

where $\mathbf{e}_1 = (1, 0, 0, \dots, 0)^T$ (see the notation in (10) once more).

The proof is by mathematical induction.

This leads us to

$$\mathbf{e}^A = (E, A, A^2, \dots, A^{n-1}) \mathbf{e}^L \mathbf{e}_1. \quad (14)$$

That is, the calculation of e^A is reduced to that of e^L . We must again calculate the exponential ! What is interesting ? In this case, we can make L diagonal completely because it is simple enough ⁴.

Let us solve the equation(s)

$$L|\alpha_j\rangle = \alpha_j|\alpha_j\rangle \quad \text{for } j = 1 \sim n, \quad (15)$$

which is easily obtained to be

$$|\alpha\rangle = \begin{pmatrix} p_{n-1} + p_{n-2}\alpha + \cdots + p_1\alpha^{n-2} + \alpha^{n-1} \\ p_{n-2} + p_{n-3}\alpha + \cdots + p_1\alpha^{n-3} + \alpha^{n-2} \\ \vdots \\ p_1 + \alpha \\ 1 \end{pmatrix}, \quad (16)$$

where $\alpha = \alpha_1, \alpha_2, \dots, \alpha_n$ for simplicity. It is of course $(\alpha_i|\alpha_j) \neq \delta_{ij}$. Then we have

$$U_L = (|\alpha_1\rangle, |\alpha_2\rangle, \dots, |\alpha_n\rangle) \in GL(n; \mathbf{C}) \quad (17)$$

and

$$L = U_L D_A U_L^{-1} \implies e^L = U_L e^{D_A} U_L^{-1}. \quad (18)$$

At first sight, it seems difficult to calculate U_L^{-1} . However, we have a simple decomposition of U_L like

$$U_L = P_L Q_L \quad (19)$$

where P_L and Q_L are given respectively as

⁴There is another diagonalization (communicated by T. Suzuki)

$$P_L = \begin{pmatrix} p_{n-1} & p_{n-2} & p_{n-3} & \cdots & p_1 & 1 \\ p_{n-2} & p_{n-3} & \cdots & p_1 & 1 & 0 \\ p_{n-3} & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & p_1 & \ddots & 0 & 0 & 0 \\ p_1 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \quad (20)$$

$$Q_L = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_{n-1}^2 & \alpha_n^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_1^{n-2} & \alpha_2^{n-2} & \cdots & \alpha_{n-1}^{n-2} & \alpha_n^{n-2} \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_{n-1}^{n-1} & \alpha_n^{n-1} \end{pmatrix}. \quad (21)$$

Therefore $U_L^{-1} = Q_L^{-1}P_L^{-1}$. Must we calculate both Q_L^{-1} and P_L^{-1} again ? From (14) it is just $U_L^{-1}\mathbf{e}_1$ not U_L^{-1} itself that we must calculate. It is easy to see

$$U_L^{-1}\mathbf{e}_1 = Q_L^{-1}P_L^{-1}\mathbf{e}_1 = Q_L^{-1}\mathbf{e}_n \quad (22)$$

where $\mathbf{e}_n = (0, 0, \dots, 0, 1)^T$ and $Q_L^{-1}\mathbf{e}_n$ is given as

$$Q_L^{-1}\mathbf{e}_n = \frac{1}{|Q_L|} \begin{pmatrix} \text{the cofactor of } \alpha_1^{n-1} \\ \text{the cofactor of } \alpha_2^{n-1} \\ \vdots \\ \text{the cofactor of } \alpha_{n-1}^{n-1} \\ \text{the cofactor of } \alpha_n^{n-1} \end{pmatrix} = (-1)^{n+1} \begin{pmatrix} \frac{1}{\prod_{j=1, j \neq 1}^n (\alpha_j - \alpha_1)} \\ \frac{1}{\prod_{j=1, j \neq 2}^n (\alpha_j - \alpha_2)} \\ \vdots \\ \frac{1}{\prod_{j=1, j \neq n-1}^n (\alpha_j - \alpha_{n-1})} \\ \frac{1}{\prod_{j=1, j \neq n}^n (\alpha_j - \alpha_n)} \end{pmatrix}. \quad (23)$$

Moreover, we can determine U_L completely. From (5) we define

$$(p_j)_k = p_j - \{\text{all the terms containing } \alpha_k\} \quad \text{for } 1 \leq k \leq n. \quad (24)$$

For example,

$$\begin{aligned} (p_1)_1 &= -(\alpha_2 + \cdots + \alpha_n), & (p_{n-1})_1 &= (-1)^{n-1} \alpha_2 \cdots \alpha_n, \\ (p_1)_2 &= -(\alpha_1 + \alpha_3 + \cdots + \alpha_n), & (p_{n-1})_2 &= (-1)^{n-1} \alpha_1 \alpha_3 \cdots \alpha_n. \end{aligned}$$

Then

$$U_L = (|\alpha_1\rangle, |\alpha_2\rangle, \dots, |\alpha_n\rangle) = \begin{pmatrix} (p_{n-1})_1 & (p_{n-1})_2 & \cdots & (p_{n-1})_{n-1} & (p_{n-1})_n \\ (p_{n-2})_1 & (p_{n-2})_2 & \cdots & (p_{n-2})_{n-1} & (p_{n-2})_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (p_1)_1 & (p_1)_2 & \cdots & (p_1)_{n-1} & (p_1)_n \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}. \quad (25)$$

Noting

$$\begin{aligned} e^L \mathbf{e}_1 &= U_L e^{D_A} U_L^{-1} \mathbf{e}_1 \\ &= (-1)^{n+1} \begin{pmatrix} (p_{n-1})_1 & (p_{n-1})_2 & \cdots & (p_{n-1})_{n-1} & (p_{n-1})_n \\ (p_{n-2})_1 & (p_{n-2})_2 & \cdots & (p_{n-2})_{n-1} & (p_{n-2})_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (p_1)_1 & (p_1)_2 & \cdots & (p_1)_{n-1} & (p_1)_n \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{e^{\alpha_1}}{\prod_{j=1, j \neq 1}^n (\alpha_j - \alpha_1)} \\ \frac{e^{\alpha_2}}{\prod_{j=1, j \neq 2}^n (\alpha_j - \alpha_2)} \\ \vdots \\ \frac{e^{\alpha_{n-1}}}{\prod_{j=1, j \neq n-1}^n (\alpha_j - \alpha_{n-1})} \\ \frac{e^{\alpha_n}}{\prod_{j=1, j \neq n}^n (\alpha_j - \alpha_n)} \end{pmatrix} \end{aligned} \quad (26)$$

we finally obtain

$$e^A = (E, A, A^2, \dots, A^{n-1}) e^L \mathbf{e}_1 \equiv f_0 E + f_1 A + f_2 A^2 + \cdots + f_{n-1} A^{n-1}$$

with

$$f_l = (-1)^{n+1} \sum_{k=1}^n \frac{(p_{n-l-1})_k e^{\alpha_k}}{\prod_{j=1, j \neq k}^n (\alpha_j - \alpha_k)} \quad 0 \leq l \leq n-1 \quad (27)$$

in a complete manner ⁵.

That is, our calculation is based on only simple operations like the powers of matrices or sum of them, etc, which is easily performed by computer.

Let us list some important examples (the case of $n = 3$ and 4).

⁵The result is also obtained by the spectral decomposition of matrices (communicated by T. Suzuki). However, it seems to us that the method is much elementary. The result has not been written in standard textbooks in Linear Algebra as far as we know

$n = 3$ For

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in M(3; \mathbf{C}) \quad (28)$$

we have

$$\begin{aligned} 0 &= |\lambda E - A| \\ &= \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32})\lambda - \\ &\quad (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}). \end{aligned} \quad (29)$$

The Cardano formula (see for example [5]) gives three solutions $\{\alpha_1, \alpha_2, \alpha_3\}$. Then

$$p_1 = -(\alpha_1 + \alpha_2 + \alpha_3), \quad p_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \quad p_3 = -\alpha_1\alpha_2\alpha_3 \quad (30)$$

and

$$L = \begin{pmatrix} 0 & 0 & -p_3 \\ 1 & 0 & -p_2 \\ 0 & 1 & -p_1 \end{pmatrix} = U_L D_A U_L^{-1} \quad (31)$$

and

$$\begin{aligned} U_L &= \begin{pmatrix} p_2 + p_1\alpha_1 + \alpha_1^2 & p_2 + p_1\alpha_2 + \alpha_2^2 & p_2 + p_1\alpha_3 + \alpha_3^2 \\ p_1 + \alpha_1 & p_1 + \alpha_2 & p_1 + \alpha_3 \\ 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_2\alpha_3 & \alpha_1\alpha_3 & \alpha_1\alpha_2 \\ -(\alpha_2 + \alpha_3) & -(\alpha_1 + \alpha_3) & -(\alpha_1 + \alpha_2) \\ 1 & 1 & 1 \end{pmatrix} \end{aligned} \quad (32)$$

and

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = e^L \mathbf{e}_1 = \begin{pmatrix} \alpha_2\alpha_3 & \alpha_1\alpha_3 & \alpha_1\alpha_2 \\ -(\alpha_2 + \alpha_3) & -(\alpha_1 + \alpha_3) & -(\alpha_1 + \alpha_2) \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{e^{\alpha_1}}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)} \\ \frac{e^{\alpha_2}}{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2)} \\ \frac{e^{\alpha_3}}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)} \end{pmatrix}. \quad (33)$$

Therefore we have finally

$$e^A = f_0 E + f_1 A + f_2 A^2 \quad (34)$$

with

$$f_0 = \frac{\alpha_2 \alpha_3 e^{\alpha_1}}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)} + \frac{\alpha_1 \alpha_3 e^{\alpha_2}}{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2)} + \frac{\alpha_1 \alpha_2 e^{\alpha_3}}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)}, \quad (35)$$

$$f_1 = -\frac{(\alpha_2 + \alpha_3)e^{\alpha_1}}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)} - \frac{(\alpha_1 + \alpha_3)e^{\alpha_2}}{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2)} - \frac{(\alpha_1 + \alpha_2)e^{\alpha_3}}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)}, \quad (36)$$

$$f_2 = \frac{e^{\alpha_1}}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)} + \frac{e^{\alpha_2}}{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2)} + \frac{e^{\alpha_3}}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)}. \quad (37)$$

$n = 4$ For

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in M(4; \mathbb{C}) \quad (38)$$

we have

$$\begin{aligned} 0 &= |\lambda E - A| \\ &= \lambda^4 - (a_{11} + a_{22} + a_{33} + a_{44})\lambda^3 + (a_{11}a_{22} + a_{11}a_{33} + a_{11}a_{44} + a_{22}a_{33} \\ &\quad + a_{22}a_{44} + a_{33}a_{44} - a_{12}a_{21} - a_{13}a_{31} - a_{14}a_{41} - a_{23}a_{32} - a_{24}a_{42} - a_{34}a_{43})\lambda^2 \\ &\quad - (a_{11}a_{22}a_{33} + a_{11}a_{22}a_{44} + a_{11}a_{33}a_{44} + a_{22}a_{33}a_{44} + a_{12}a_{23}a_{31} + a_{12}a_{24}a_{41} \\ &\quad + a_{13}a_{21}a_{32} + a_{13}a_{34}a_{41} + a_{14}a_{21}a_{42} + a_{14}a_{31}a_{43} + a_{23}a_{34}a_{42} + a_{24}a_{32}a_{43} \\ &\quad - a_{11}a_{23}a_{32} - a_{11}a_{24}a_{42} - a_{11}a_{34}a_{43} - a_{12}a_{21}a_{33} - a_{12}a_{21}a_{44} - a_{13}a_{22}a_{31} \\ &\quad - a_{13}a_{31}a_{44} - a_{14}a_{22}a_{41} - a_{14}a_{33}a_{41} - a_{22}a_{34}a_{43} - a_{23}a_{32}a_{44} - a_{24}a_{33}a_{42})\lambda \\ &\quad + \det(A) \end{aligned} \quad (39)$$

where $\det(A)$ is omitted, see for example [4]. The Ferrari formula or Euler one (see [5])

gives four solutions $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ ⁶. Then

$$\begin{aligned} p_1 &= -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \quad p_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4, \\ p_3 &= -(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4), \quad p_4 = \alpha_1\alpha_2\alpha_3\alpha_4 \end{aligned} \quad (40)$$

and

$$L = \begin{pmatrix} 0 & 0 & 0 & -p_4 \\ 1 & 0 & 0 & -p_3 \\ 0 & 1 & 0 & -p_2 \\ 0 & 0 & 1 & -p_1 \end{pmatrix} = U_L D_A U_L^{-1} \quad (41)$$

and

$$\begin{aligned} U_L &= \begin{pmatrix} p_3 + p_2\alpha_1 + p_1\alpha_1^2 + \alpha_1^3 & p_3 + p_2\alpha_2 + p_1\alpha_2^2 + \alpha_2^3 & p_3 + p_2\alpha_3 + p_1\alpha_3^2 + \alpha_3^3 & p_3 + p_2\alpha_4 + p_1\alpha_4^2 + \alpha_4^3 \\ p_2 + p_1\alpha_1 + \alpha_1^2 & p_2 + p_1\alpha_2 + \alpha_2^2 & p_2 + p_1\alpha_3 + \alpha_3^2 & p_2 + p_1\alpha_4 + \alpha_4^2 \\ p_1 + \alpha_1 & p_1 + \alpha_2 & p_1 + \alpha_3 & p_1 + \alpha_4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\alpha_2\alpha_3\alpha_4 & -\alpha_1\alpha_3\alpha_4 & -\alpha_1\alpha_2\alpha_4 & -\alpha_1\alpha_2\alpha_3 \\ \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 & \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_3\alpha_4 & \alpha_1\alpha_2 + \alpha_1\alpha_4 + \alpha_2\alpha_4 & \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 \\ -(\alpha_2 + \alpha_3 + \alpha_4) & -(\alpha_1 + \alpha_3 + \alpha_4) & -(\alpha_1 + \alpha_2 + \alpha_4) & -(\alpha_1 + \alpha_2 + \alpha_3) \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{aligned} \quad (42)$$

and

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = e^L \mathbf{e}_1 = -U_L \begin{pmatrix} \frac{e^{\alpha_1}}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)(\alpha_4 - \alpha_1)} \\ \frac{e^{\alpha_2}}{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2)(\alpha_4 - \alpha_2)} \\ \frac{e^{\alpha_3}}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)(\alpha_4 - \alpha_3)} \\ \frac{e^{\alpha_4}}{(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_4)(\alpha_3 - \alpha_4)} \end{pmatrix}. \quad (43)$$

Therefore we have finally

$$e^A = f_0 E + f_1 A + f_2 A^2 + f_3 A^3 \quad (44)$$

⁶They are of course too complicated

with

$$f_0 = \frac{\alpha_2\alpha_3\alpha_4e^{\alpha_1}}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)(\alpha_4-\alpha_1)} + \frac{\alpha_1\alpha_3\alpha_4e^{\alpha_2}}{(\alpha_1-\alpha_2)(\alpha_3-\alpha_2)(\alpha_4-\alpha_2)} + \frac{\alpha_1\alpha_2\alpha_4e^{\alpha_3}}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)(\alpha_4-\alpha_3)} + \frac{\alpha_1\alpha_2\alpha_3e^{\alpha_4}}{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4)}, \quad (45)$$

$$f_1 = -\frac{(\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4)e^{\alpha_1}}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)(\alpha_4-\alpha_1)} - \frac{(\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_3\alpha_4)e^{\alpha_2}}{(\alpha_1-\alpha_2)(\alpha_3-\alpha_2)(\alpha_4-\alpha_2)} - \frac{(\alpha_1\alpha_2+\alpha_1\alpha_4+\alpha_2\alpha_4)e^{\alpha_3}}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)(\alpha_4-\alpha_3)} - \frac{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)e^{\alpha_4}}{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4)}, \quad (46)$$

$$f_2 = \frac{(\alpha_2+\alpha_3+\alpha_4)e^{\alpha_1}}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)(\alpha_4-\alpha_1)} + \frac{(\alpha_1+\alpha_3+\alpha_4)e^{\alpha_2}}{(\alpha_1-\alpha_2)(\alpha_3-\alpha_2)(\alpha_4-\alpha_2)} + \frac{(\alpha_1+\alpha_2+\alpha_4)e^{\alpha_3}}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)(\alpha_4-\alpha_3)} + \frac{(\alpha_1+\alpha_2+\alpha_3)e^{\alpha_4}}{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4)}, \quad (47)$$

$$f_3 = -\frac{e^{\alpha_1}}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)(\alpha_4-\alpha_1)} - \frac{e^{\alpha_2}}{(\alpha_1-\alpha_2)(\alpha_3-\alpha_2)(\alpha_4-\alpha_2)} - \frac{e^{\alpha_3}}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)(\alpha_4-\alpha_3)} - \frac{e^{\alpha_4}}{(\alpha_1-\alpha_4)(\alpha_2-\alpha_4)(\alpha_3-\alpha_4)}. \quad (48)$$

A comment is in order.

(1) At first sight, the eigenvalues in our formula (44) appear to be different all. However, it is not true. Let us explain this with a simple example.

For $A \in M(2; \mathbf{C})$

$$0 = |\lambda E - A| = \lambda^2 + p_1\lambda + p_2 = (\lambda - \alpha)(\lambda - \beta),$$

which gives $p_1 = -(\alpha + \beta)$, $p_2 = \alpha\beta$ and

$$L = \begin{pmatrix} 0 & -p_2 \\ 1 & -p_1 \end{pmatrix}.$$

Then the formula is

$$e^A = (E, A)e^L \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In the following let us consider two cases.

(I) $\alpha \neq \beta$:

In this case, $L = U_L \begin{pmatrix} \beta & \\ & \alpha \end{pmatrix} U_L^{-1}$ with

$$U_L = \begin{pmatrix} -\alpha & -\beta \\ 1 & 1 \end{pmatrix} \Rightarrow U_L^{-1} = \frac{1}{\beta - \alpha} \begin{pmatrix} 1 & \beta \\ -1 & -\alpha \end{pmatrix}$$

and it is easy to see

$$e^L \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\beta e^\alpha - \alpha e^\beta}{\beta - \alpha} \\ \frac{e^\beta - e^\alpha}{\beta - \alpha} \end{pmatrix} = \begin{pmatrix} e^\alpha - \alpha \frac{e^\beta - e^\alpha}{\beta - \alpha} \\ \frac{e^\beta - e^\alpha}{\beta - \alpha} \end{pmatrix}. \quad (49)$$

(II) $\alpha = \beta$:

In this case, L is

$$L = \begin{pmatrix} 0 & -p_2 \\ 1 & -p_1 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha^2 \\ 1 & 2\alpha \end{pmatrix},$$

we cannot diagonalize L , so we must use the method of Jordan canonical form. Since this method is well-known (see [4]) the details are omitted. If we set

$$U_L = \begin{pmatrix} -\alpha & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow U_L^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}$$

then it is easy to see

$$U_L \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} U_L^{-1} = \begin{pmatrix} 0 & -\alpha^2 \\ 1 & 2\alpha \end{pmatrix} = L.$$

Therefore

$$e^L = U_L \exp \left\{ \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \right\} U_L^{-1} = U_L \begin{pmatrix} e^\alpha & e^\alpha \\ 0 & e^\alpha \end{pmatrix} U_L^{-1},$$

we have

$$e^L \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} (1 - \alpha)e^\alpha \\ e^\alpha \end{pmatrix}. \quad (50)$$

Comparing (50) with (49) it is clear that $\lim_{\beta \rightarrow \alpha} (49) = (50)$. That is, our method covers all special cases by taking some appropriate limit.

To obtain the formula including multiplicities of eigenvalues is a good exercise. We leave it to readers.

(2) In realistic problems we must calculate e^{tA} in place of e^A . The result is changed simply into

$$e^{tA} = f_0 E + f_1 A + f_2 A^2 + \cdots + f_{n-1} A^{n-1} \quad (51)$$

with

$$f_l = (-1)^{n+1} \sum_{k=1}^n \frac{(p_{n-l-1})_k e^{t\alpha_k}}{\prod_{j=1, j \neq k}^n (\alpha_j - \alpha_k)} \quad 0 \leq l \leq n-1. \quad (52)$$

We leave the check to readers.

(3) We comment on a generalization of the result. Let F be an entire function on \mathbf{C} . Then the result is changed simply into

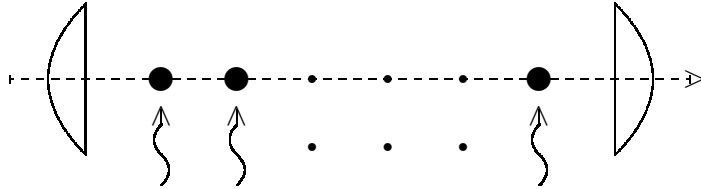
$$F(A) = f_0 E + f_1 A + f_2 A^2 + \cdots + f_{n-1} A^{n-1} \quad (53)$$

with

$$f_l = (-1)^{n+1} \sum_{k=1}^n \frac{(p_{n-l-1})_k F(\alpha_k)}{\prod_{j=1, j \neq k}^n (\alpha_j - \alpha_k)} \quad 0 \leq l \leq n-1. \quad (54)$$

We leave the check to readers.

We conclude this paper by stating our motivation. We are studying a quantum computation based on Cavity QED whose image is



The general setting for a quantum computation based on Cavity QED :

the dotted line means a single photon inserted in the cavity and

all curves mean external laser fields subjected to atoms

See [6], [7] in detail. It is usually based on two level system of atoms. However, to take a multi level system of them into consideration may be better from the view point of decoherence which is a severe problem in quantum computation. To develop quantum circuits (see for example [3], [8] and [9]) we often encounter the problem to calculate the exponential of matrices in an explicit manner, which was very difficult (for at least Fujii).

Since we have somewhat overcome this difficulty it must be possible to reconsider quantum circuits in the multi level system.

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